Post-radiation evolution of black holes

G.A. Vilkovisky

Lebedev Physical Institute, and Lebedev Research Center in Physics,

Leninsky Prospect 53, 119991 Moscow, Russia.

E-mail: vilkov@sci.lebedev.ru

Abstract

The expectation-value equations for the collapse of a macroscopic, spherically symmetric, and uncharged body are integrated up to the limit of validity of semi-classical theory. The collapse finishes with a true stable black hole of the mass microscopically exceeding the vacuum-induced charge. The apparent horizon is almost closed. The most important feature of the solution is the presence of an irremovable Cauchy horizon.

The present paper is a sequel of Ref. [1] in which the expectation-value problem is considered for the gravitational collapse of a spherically symmetric uncharged source having a compact spatial support and a macroscopic ADM mass M. The evolution equations for the metric have been integrated in Ref. [1] only up to the instant of retarded time at which the Hawking radiation terminates, but the problem of the further integration was not that the limit of validity of semiclassical theory was reached. Rather, some of the techniques used in Ref. [1] ceased working. Here these difficulties are surmounted, and the solution is pushed to the limit of validity of semiclassical theory.

The notation in the present paper is the same as in Ref. [1] except for λ in Eq. (25) below. Specifically, $v = \text{const.}$ and $u = \text{const.}$ are the equations of the radial past and future light cones, and

$$A = 4\pi r^2 \left(T_{\text{vac}}^{\mu\nu} \nabla_\mu u \nabla_\nu u \right) \frac{(\nabla r, \nabla v)^2}{(\nabla u, \nabla v)^2} \,, \tag{1}$$

$$D = 4\pi r^2 \left(T_{\text{vac}}^{\mu\nu} \nabla_\mu v \nabla_\nu v \right) \frac{1}{(\nabla r, \nabla v)^2} \,, \tag{2}$$

$$T_1 = 4\pi r^2 \left(T_{\text{vac}}^{\mu\nu} \nabla_\mu u \nabla_\nu v \right) \frac{2}{(\nabla u, \nabla v)} \,, \tag{3}$$

$$T_1 + T_2 = 4\pi r^2 T_{\text{vac}}^{\mu\nu} g_{\mu\nu} \,, \tag{4}$$

$$B = 1 - (\nabla r)^2 + T_1 = r \triangle r \,. \tag{5}$$

Here $T_{\text{vac}}^{\mu\nu}$ is the energy-momentum tensor of the in-vacuum, and $\triangle$ is the D'Alembert operator in the Lorentzian subspace of a spherically symmetric spacetime. Below, both v and u are normalized so as to be the proper time of an observer at infinity. $\mathcal{O}$ is the notation for a quantity that vanishes as $\mu/M \to 0$, and μ is the Planckian mass (1 in the absolute units). An inequality of the form $X > |\mathcal{O}|$ assumes any $\mathcal{O}$ and signifies that X is a macroscopic quantity.

As discussed in Ref. [1], in the semiclassical region [1] the expectation value of the metric satisfies the equations

$$T_1 + T_2 = \mathcal{O} \,, \qquad A = \mathcal{O} \,. \tag{6}$$

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It has been shown in Ref. [1] that, as far as the solution obtained in this reference extends, it satisfies also the equations

$$T_1 + T_2 = \mathcal{O} , \qquad D = \mathcal{O} . \tag{7}$$

A necessary condition for the equivalence of Eqs. (6) and (7) is that the data functions be slowly varying with respect to the time of an observer at infinity. No signs of a violation of this condition are observed in the solution. The obstacle encountered in Ref. [1] was the appearance in the solution of a line $B = 0$. The appearance of this line at sufficiently late u is directly related to the vanishing of the radiation temperature. The idea of the present work is to use Eqs. (7) to get across $B = 0$.

Both sets of equations, (6) and (7), are valid globally but neither of them can be used to obtain the solution globally. The solution can be obtained in two complementary regions. Eqs. (6) can be solved in the region

$$(\nabla r)^2 > O\left(\sqrt{|A|}\right) \tag{8}$$

called in [1] the region of weak field, and the solution is [1]

$$(\nabla r)^2 = 1 - \frac{2\mathcal{M}(u)}{r} + \frac{Q^2(u)}{r^2} , \tag{9}$$

$$(\nabla r, \nabla u) = -1 + \mathcal{O} , \tag{10}$$

$$B = \frac{2}{r}\left(\mathcal{M}(u) - \frac{Q^2(u)}{r}\right) , \tag{11}$$

$$D = \frac{4}{((\nabla r)^2)^2}\left(-\frac{d\mathcal{M}(u)}{du} + \frac{1}{2r}\frac{dQ^2(u)}{du}\right) \tag{12}$$

where $\mathcal{M}(u)$ and $Q^2(u)$ are arbitrary data functions.

Eqs. (7) can be solved in the region

$$r < \frac{1}{|\mathcal{O}|} \tag{13}$$

which covers the region of strong field

$$\left((\nabla r)^2 < 0\right)\bigcup\left((\nabla r)^2 = |\mathcal{O}|\right) . \tag{14}$$

The solution is

$$(\nabla r)^2 = 1 - \frac{2m(v)}{r} + \frac{q^2(v)}{r^2} , \tag{15}$$

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$$B = \frac{2}{r}\left(m(v) - \frac{q^2(v)}{r}\right) , \tag{16}$$

$$A = \frac{dm(v)}{dv} - \frac{1}{2r}\frac{dq^2(v)}{dv} , \tag{17}$$

and the equation

$$(\nabla r, \nabla v) = 1 + \mathcal{O} \tag{18}$$

holds globally [1]. Here $m(v)$ and $q^2(v)$ are arbitrary data functions. It is explained in Ref. [1] why the regions of validity of the two solutions are different.

The two metrics can be joined together on any line L in the region of overlap. For L one can take a line of constant $(\nabla r)^2$ with the value of $(\nabla r)^2$ small but satisfying (8). The equation of this line in the coordinates u, v can be calculated, and it is

$$L: \qquad u = u_L(v) , \qquad \frac{du_L(v)}{dv} = 1 + \mathcal{O} . \tag{19}$$

To locate the line L, one then only needs to specify any point through which it passes. Such a point is known [1]:

$$u = u_0 , \qquad v = v_{\text{crit}} = v_0 + 4M \ln \frac{M^2}{\mu^2} + O(M) \tag{20}$$

where (u_0, v_0) is the point at which the apparent horizon $(\nabla r)^2 = 0$ is null [1]. The junction conditions are

$$m(v) = \mathcal{M}\left(u_L(v)\right) , \qquad q^2(v) = Q^2\left(u_L(v)\right) . \tag{21}$$

The initial values of the data functions are

$$m(v_0) = M , \qquad q^2(v_0) = 0 . \tag{22}$$

The v_{crit} in Eq. (20) is the bound set for v by the correspondence principle [1]. At $v \leq v_{\text{crit}}$, the classical geometry is valid, and the functions $m(v)$ and $q^2(v)$ differ from their values in (22) microscopically.

The data functions are expressed through the metric in a self-consistent manner. The metric should be transformed into the null coordinates: $r = r(u, v)$, and the following function should be calculated:

$$\kappa(u) = -\frac{d}{du} \ln \left(-2 \frac{\partial r(u, v)}{\partial u}\right)\bigg|_{v=v_0} . \tag{23}$$

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The argument of ln in this expression is the red-shift factor. The equations for the data functions as obtained in Ref. [1] are of the form

$$-\frac{d\mathcal{M}(u)}{du} = \lambda\kappa^2(u) , \qquad \frac{dQ^2(u)}{du} = 2\lambda\kappa(u) , \tag{24}$$

$$\lambda = \frac{\mu^2}{48\pi} . \tag{25}$$

The numerical coefficient in (25) is calculated for the vacuum of the massless spin-0 particles, and in (24) only the contribution of the quantum s-mode is retained. The equation for $\mathcal{M}(u)$ has also a contribution of higher-l modes [1] but it is by an order of magnitude smaller.

By derivation [1], Eqs. (24) are valid in the range of u in which two conditions are fulfilled:

$$\int\limits_{-\infty}^{u} du\,\kappa \gg 1 , \tag{26}$$

$$\frac{d}{du}\frac{1}{\kappa} \ll 1 . \tag{27}$$

Eq. (26) is a condition that the red shift is large. Eq. (27) holds when κ is a macroscopic quantity, and $d\kappa/du$ is a microscopic quantity. Both conditions are fulfilled indeed during the entire radiation stage of the evolution. At the post-radiation stage, condition (26) remains valid but condition (27) does not. The point is that $d\kappa/du$ remains a microscopic quantity but $\kappa \to 0$. As κ^2 reaches the value of $d\kappa/du$, the quantity in (27) becomes $O(1)$. Removal of condition (27) is another problem that needs to be solved in the present work.

Condition (27) can be removed but the data equations get modified. The refined equations are of the form

$$-\frac{d\mathcal{M}(u)}{du} = \lambda\left(\kappa^2(u) + 2\frac{d\kappa(u)}{du} - \frac{d\widetilde{\kappa}(u)}{du}\right) , \tag{28}$$

$$\frac{dQ^2(u)}{du} = 2\lambda\widetilde{\kappa}(u) \tag{29}$$

where the function $\widetilde{\kappa}(u)$ is expressed through $\kappa(u)$ by the differential equation

$$\frac{d\widetilde{\kappa}}{du} = \widetilde{\kappa}\left(\widetilde{\kappa} - \kappa\right) \tag{30}$$

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with the boundary condition

$$\frac{1}{\widetilde{\kappa}(u)} \exp\left(-\int\limits_{-\infty}^{u} du\,\kappa\right) \to 0 \quad \text{as} \quad \int\limits_{-\infty}^{u} du\,\kappa \to \infty \ . \tag{31}$$

Eq. (26) remains the only condition of validity of these equations. In the range of u in which $\kappa(u)$ satisfies also condition (27), the solution of Eqs. (30), (31) is a series

$$\frac{1}{\widetilde{\kappa}} = \frac{1}{\kappa}\left(1 + \frac{d}{du}\frac{1}{\kappa} + \frac{d}{du}\frac{1}{\kappa}\frac{d}{du}\frac{1}{\kappa} + \cdots\right) \tag{32}$$

in which all terms with derivatives are negligible. Then $\widetilde{\kappa} = \kappa$, and one recovers Eqs. (24). The modification of the data equations is the most important fact for the presently considered problem. Therefore, its derivation is sketched in the Appendix below. The principal consequence of the modified equations is that $\widetilde{\kappa}(u)$ cannot turn into zero at a finite value of u.

The equations above make a closed system, and I go over to their solution. Consider an outgoing light ray $u = $ const. that in the region of strong field does not meet with microscopic B, and calculate the minimum of $(\nabla r)^2$ on this ray. The equation of the minimum is [1]

$$B(\nabla r)^2 = 4A \ . \tag{33}$$

Since $A = -|\mathcal{O}|$ and, on the ray considered, $B > |\mathcal{O}|$, one finds that $(\nabla r)^2$ at the minimum is $\mathcal{O}$ and is negative. Therefore, $(\nabla r)^2$ is $\mathcal{O}$ in the entire subregion of strong field foliated by such light rays. It then follows from (15) that, in this subregion,

$$r = r_{\text{AH}}^{+}(v) + \mathcal{O} \ , \tag{34}$$

i.e., r is a function only of v. Here and below, $\text{AH}^{\pm}$ are the two solutions for the apparent horizon given by Eq. (15):

$$r_{\text{AH}}^{\pm}(v) = m(v) \pm \sqrt{m^2(v) - q^2(v)} \ . \tag{35}$$

Since r is a function only of v, so are B and A in (16) and (17). Then consider the solution of Eq. (5):

$$-\ln\left(-2\frac{\partial r(u,v)}{\partial u}\right) = \int\limits_{v}^{\infty} dv\, \frac{B}{2r}\bigg|_{u=\text{const.}} = \int\limits_{v_L(u)}^{\infty} dv\, \frac{B}{2r}\bigg|_{u=\text{const.}} + \int\limits_{v}^{v_L(u)} dv\, \frac{B}{2r}\bigg|_{u=\text{const.}} \ . \tag{36}$$

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A use of (34) enables one to calculate the strong-field contribution in this integral:

$$\int\limits_{v}^{v_L(u)} dv \, \frac{B}{2r}\bigg|_{u=\text{const.}} = \int\limits_{v}^{v_L(u)} dv \, \frac{B_{\text{AH}}^+(v)}{2r_{\text{AH}}^+(v)} \,, \tag{37}$$

$$\frac{d}{du} \int\limits_{v}^{v_L(u)} dv \, \frac{B}{2r}\bigg|_{u=\text{const.}} = \frac{B_{\text{AH}}^+\left(v_L(u)\right)}{2r_{\text{AH}}^+\left(v_L(u)\right)} \,. \tag{38}$$

In this way one recovers the strong-field metric of Ref. [1].

The weak-field contribution in (36) is readily obtained with the aid of Eq. (10):

$$\int\limits_{v_L(u)}^{\infty} dv \, \frac{B}{2r}\bigg|_{u=\text{const.}} = -\ln(\nabla r)^2 \big|_{v=v_L(u)} \,, \tag{39}$$

$$\frac{d}{du} \int\limits_{v_L(u)}^{\infty} dv \, \frac{B}{2r}\bigg|_{u=\text{const.}} = \frac{1}{2r}\left(\frac{4A}{(\nabla r)^2} - D(\nabla r)^2\right)\bigg|_{L} = \mathcal{O} \,. \tag{40}$$

Combining (38) and (40) gives κ. Up to $\mathcal{O}$,

$$\begin{aligned}
\kappa(u) &= \frac{B_{\text{AH}}^+\left(v_L(u)\right)}{2r_{\text{AH}}^+\left(v_L(u)\right)} = \frac{\sqrt{m^2(v)-q^2(v)}}{\left(m(v)+\sqrt{m^2(v)-q^2(v)}\right)^2}\bigg|_{v=v_L(u)} \\
&= \frac{\sqrt{\mathcal{M}^2(u)-Q^2(u)}}{\left(\mathcal{M}(u)+\sqrt{\mathcal{M}^2(u)-Q^2(u)}\right)^2} \,. \tag{41}
\end{aligned}$$

With this expression, Eqs. (24) close, and one recovers completely the results of Ref. [1].

However, one obtains also the limitations on the validity of these results. The region of their validity is bounded by the latest light ray $u = \text{const.}$ that, in the strong field, does not meet with small B. It can be shown that this upper bound in u is the same as the one put by condition (27):

$$\kappa(u) > O(\sqrt{\lambda}) \,, \qquad \mathcal{M}^2(u) - Q^2(u) > |O(\lambda)| \,. \tag{42}$$

By the junction conditions, the respective subregion of strong field is bounded also from above in v:

$$m^2(v) - q^2(v) > |O(\lambda)| \,. \tag{43}$$

By exploiting Eq. (7) we have removed one of these limitations. Namely, under condition (43), the metric in (15)–(18) continues the strong-field solution along the lines $v = \text{const.}$

across $B = 0$ and AH^- down to $r = \mathcal{O}$, i.e., to the limit of validity of semiclassical theory. The significance of the line $B = 0$ is that, on it, $(\nabla r)^2$ has a minimum along the incoming light rays $v = \text{const.}$, which enables these rays to cross the apparent horizon twice. The main features of the strong-field solution are shown in Fig. 1 in the coordinates of a falling observer. The subregion where the metric of Ref. [1] is valid is shown foliated with the light rays $u = \text{const.}$ In the extended metric, Eq. (33) has three solutions for each v shown with broken lines. Every outgoing ray $u = \text{const.}$ inside the apparent horizon crosses one of these lines.

The values of u and v corresponding to the bounds (42) and (43) (call them u_1 and v_1) are obtained in Ref. [1]. Up to negligible corrections,

$$u_1 - u_0 = 96\pi \frac{M^3}{\mu^2} , \qquad v_1 - v_0 = 96\pi \frac{M^3}{\mu^2} . \tag{44}$$

There remains to be considered the region $u > u_1$, $v > v_1$ marked with "?" in Fig. 1. At $u > u_1$, $\kappa(u) = O(\sqrt{\lambda})$ and, therefore,

$$-\frac{d\mathcal{M}(u)}{du} = O(\lambda^2) , \qquad \frac{dQ^2(u)}{du} = O(\lambda^{3/2}) . \tag{45}$$

It is seen that the flux of $\mathcal{M}$ is already at the two-loop level and is negligible as compared to the flux of Q^2. Therefore, at $u > u_1$, $\mathcal{M}(u)$ may be considered constant:

$$\mathcal{M}(u) = \mathcal{M}_1 = \text{const.} , \qquad u > u_1 . \tag{46}$$

The flux of Q^2 is, on the other hand, significant notwithstanding the fact that

$$\mathcal{M}_1{}^2 - Q^2(u) = O(\lambda) , \qquad u > u_1 . \tag{47}$$

The value of $\mathcal{M}_1$ is obtained in Ref. [1]. $\mathcal{M}_1$ is approximately 90% of the ADM mass M.

In the strong-field sector of the region "?" one has accordingly

$$m(v) = m_1 = \text{const.} , \quad m_1 = \mathcal{M}_1 , \quad m_1{}^2 - q^2(v) = O(\lambda) , \quad v > v_1 \tag{48}$$

and, in addition,

$$B = \mathcal{O} , \qquad (\nabla r)^2 = \mathcal{O} , \qquad r = m_1 + \mathcal{O} . \tag{49}$$

The last three conditions are equivalent in consequence of (48). As a result, the metric in (15)–(17) simplifies as follows:

$$(\nabla r)^2 = \frac{1}{4}B^2 + \frac{q^2(v) - m_1{}^2}{m_1{}^2} \ , \tag{50}$$

$$B = 2\frac{r - m_1}{m_1} \ , \qquad A = -\frac{1}{2m_1}\frac{dq^2(v)}{dv} \ . \tag{51}$$

In order to transform this metric into the null coordinates, differentiate Eq. (5) with B inserted from (16), and next use (48) and (49):

$$\frac{\partial^2}{\partial u \partial v}\ln\left(-\frac{\partial r(u,v)}{\partial u}\right) = \left(3\frac{q^2(v)}{r^4} - 2\frac{m(v)}{r^3}\right)\frac{\partial r(u,v)}{\partial u} = \left(\frac{1}{m_1{}^2} + \mathcal{O}\right)\frac{\partial r(u,v)}{\partial u} \ . \tag{52}$$

In view of Eq. (18), this relation can also be written in the form

$$\frac{\partial^2}{\partial u \partial v}\ln\left(-\frac{1}{(\nabla u, \nabla v)}\right) = \left(\frac{1}{m_1{}^2} + \mathcal{O}\right)\frac{1}{(\nabla u, \nabla v)} \ , \tag{53}$$

and it means that, in the strong-field sector of the region "?", the Lorentzian subspace is the space of constant curvature. Eq. (52) solves as

$$\frac{\partial r(u,v)}{\partial u} = -2m_1{}^2\frac{f'(v)g'(u)}{(f(v) - g(u))^2} \ , \tag{54}$$

$$\frac{r - m_1}{m_1} = \frac{1}{2}B = m_1\frac{f''(v)}{f'(v)} - 2m_1\frac{f'(v)}{f(v) - g(u)} \tag{55}$$

with the functions $f(v)$ and $g(u)$ to be fixed by the choice of the coordinates v and u. The primes on these functions designate the differentiation with respect to their arguments.

The condition that fixes the normalization of u is

$$(\nabla r, \nabla u)\big|_L \equiv (\nabla r)^2\left(2\frac{\partial r(u,v)}{\partial u}\right)^{-1}\bigg|_L = -1 \ . \tag{56}$$

In a neighbourhood of the line L, $(\nabla r)^2$ should grow to reach its weak-field value (8). This suggests that the line L is

$$L: \qquad f(v) = g(u) \ . \tag{57}$$

With (57), condition (56) gets satisfied and, thereby, fixes the function $g(u)$ as

$$g(u) = f(v_L(u)) \ . \tag{58}$$

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The condition that fixes the normalization of v is (18), i.e.,

$$\frac{\partial r(u,v)}{\partial v} = \frac{1}{2}\left(\nabla r\right)^2 .$$ (59)

This gives the equation for $f(v)$

$$2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2 + \frac{m_1{}^2 - q^2(v)}{m_1{}^4} = 0 .$$ (60)

The equivalent equation for $g(u)$ is

$$2\left(\frac{g''}{g'}\right)' - \left(\frac{g''}{g'}\right)^2 + \frac{m_1{}^2 - Q^2(u)}{m_1{}^4} = 0 .$$ (61)

Both the metric in (54), (55) and the equations for f and g are invariant under the linear transformations

$$f \to \alpha f + \beta , \quad g \to \alpha g + \beta , \qquad \alpha, \beta = \text{const.}$$ (62)

and the transformation

$$f \to \frac{1}{f} , \quad g \to \frac{1}{g} .$$ (63)

These transformations make a gauge arbitrariness of the solution for f and g.

For the red-shift factor in (23) one can write

$$-2\left.\frac{\partial r(u,v)}{\partial u}\right|_{v=v_0} = -2\left.\frac{\partial r(u,v)}{\partial u}\right|_{v=v_1} \exp\left(-\int_{v_0}^{v_1} dv \left.\frac{B}{2r}\right|_{u=\text{const.}}\right) .$$ (64)

To the integral in this expression, Eq. (34) applies. Therefore, the exponential does not depend on u. When using (54) for the remaining factor, it is convenient to make the replacement

$$\frac{1}{f(v_1) - g(u)} \to g(u) ,$$ (65)

thereby fixing the gauge with respect to the transformation (63). As a result, $\kappa(u)$ is expressed through $g(u)$ in this gauge as

$$\kappa(u) = -\frac{g''(u)}{g'(u)} .$$ (66)

With this expression, Eq. (61) becomes the equation for $\kappa(u)$

$$-2\frac{d\kappa(u)}{du} - \kappa^2(u) + \frac{m_1{}^2 - Q^2(u)}{m_1{}^4} = 0 .$$ (67)

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Eqs. (29), (30), and (67) close. The closed system boils down to two coupled equations for κ and $\widetilde{\kappa}$ as functions of Q^2. Upon the scaling

$$\kappa = \sqrt{\frac{2\lambda}{m_1{}^4}}\, k \, , \quad \widetilde{\kappa} = \sqrt{\frac{2\lambda}{m_1{}^4}}\, \widetilde{k} \, , \quad \frac{Q^2 - m_1{}^2}{2\lambda} = \delta \, , \tag{68}$$

these equations take the form

$$-2\,\frac{dk}{d\delta} = \frac{k^2 + \delta}{\widetilde{k}} \, , \qquad \frac{d\widetilde{k}}{d\delta} = \widetilde{k} - k \, . \tag{69}$$

The function $Q^2(u)$ is expressed through their solution by the exact integral

$$\text{const.} + u = \sqrt{\frac{m_1{}^4}{2\lambda}} \, \frac{k^2 - 2k\widetilde{k} + \delta}{\widetilde{k}} \, , \tag{70}$$

and for the functions $g(u)$ and $f(v)$ one obtains the exact expressions

$$g(u) = \exp\left(\frac{m_1{}^2 - Q^2(u)}{2\lambda}\right) \, , \qquad f(v) = \exp\left(\frac{m_1{}^2 - q^2(v)}{2\lambda}\right) \tag{71}$$

in which the gauge arbitrariness is already completely fixed.

One boundary condition to Eqs. (69) is (31), and it implies

$$\frac{1}{\widetilde{k}} = e^{\Delta} \int_{\Delta}^{\infty} d\Delta\, e^{-\Delta}\, \frac{1}{k} \, , \qquad \Delta = \int_{-\infty}^{u} du\, \kappa \, , \qquad \frac{d\Delta}{d\delta} = \frac{k}{\widetilde{k}} \, . \tag{72}$$

The other one is that the solution should conform to (41):

$$k = \widetilde{k} = \sqrt{-\delta} \, , \qquad \delta \to -\infty \, . \tag{73}$$

It implies

$$\frac{dk}{d\delta} = -\frac{1}{2}\, e^{-\delta} \int_{-\infty}^{\delta} d\delta\, e^{\delta}\, \frac{1}{\widetilde{k}} \, . \tag{74}$$

From (74) one has $dk/d\delta < 0$. Then from (72) one has $\widetilde{k} < k$, and then from (69) one has $d\widetilde{k}/d\delta < 0$. This is sufficient to specify the solution.

The solution extends to $u = +\infty$. The Bondi charge $Q^2(u)$ grows monotonically up to the value

$$Q^2(u) = m_1{}^2 + 2\lambda\delta_0 \, , \qquad \delta_0 \leq 0 \, , \quad u \to +\infty \tag{75}$$

and the function $\widetilde{\kappa}(u)$ decreases monotonically down to zero according to the law

$$\widetilde{\kappa}(u) = \sqrt{\frac{2\lambda(-\delta_0)}{m_1{}^4}} \exp\left(-\sqrt{\frac{2\lambda(-\delta_0)}{m_1{}^4}}\, u\right) \, , \qquad \delta_0 < 0 \, , \quad u \to +\infty \, . \tag{76}$$

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The function $\kappa(u)$ varies within the limits

$$\frac{\sqrt{m_1{}^2 - Q^2(u)}}{m_1{}^2} < \kappa(u) < \frac{\sqrt{m_1{}^2 + 2\lambda - Q^2(u)}}{m_1{}^2} \tag{77}$$

and, at $u \to +\infty$, decreases down to a finite value:

$$\kappa(u) = \sqrt{\frac{2\lambda(-\delta_0)}{m_1{}^4}} + \frac{\lambda}{m_1{}^4} u \exp\left(-\sqrt{\frac{2\lambda(-\delta_0)}{m_1{}^4}}\, u\right) , \qquad \delta_0 < 0 , \quad u \to +\infty . \tag{78}$$

For $\delta_0 = 0$, the laws of decrease are

$$\widetilde{\kappa}(u) = \frac{m_1{}^4}{\lambda} \frac{3}{u^3} , \qquad \kappa(u) = \frac{3}{u} , \qquad \delta_0 = 0 , \quad u \to +\infty . \tag{79}$$

The difference between the cases $\delta_0 < 0$ and $\delta_0 = 0$ is unessential. The important point is that, by virtue of the solution, δ_0 is *necessarily nonpositive*. The $-\delta_0$ is a pure number whose specific value is immaterial.

To summarize, the Bondi charges $\mathcal{M}^2(u)$ and $Q^2(u)$ begin with the values

$$\mathcal{M}^2(u) = M^2 + \mathcal{O} , \qquad Q^2(u) = \mathcal{O} \qquad \text{at} \quad u \leq u_0 \tag{80}$$

and next draw together. They reach the level

$$\mathcal{M}^2(u) - Q^2(u) = |O(\lambda)| \tag{81}$$

for a finite time u in Eq. (44), but then they stay at this level for an infinitely long time u, their fluxes gradually decreasing down to zero.

Upon the use of (66) and (71), the expressions (54), (55) for the strong-field metric take the form

$$\frac{\partial r(u,v)}{\partial u} = -2m_1{}^2\widetilde{\kappa}(u)\widetilde{\kappa}\left(u_L(v)\right) \frac{f(v)g(u)}{(f(v) - g(u))^2} , \tag{82}$$

$$r - m_1 = -m_1{}^2\kappa\left(u_L(v)\right) + 2m_1{}^2\widetilde{\kappa}\left(u_L(v)\right) \frac{f(v)}{f(v) - g(u)} . \tag{83}$$

The data function $q^2(v)$ and hence the functions $f(v)$, $\kappa\left(u_L(v)\right)$, $\widetilde{\kappa}\left(u_L(v)\right)$ are now known for all v up to $v = \infty$. Specifically, the behaviours of these functions at $v \to \infty$ are obtained from (75)–(79) by replacing u with v, and $Q^2(u)$ with $q^2(v)$.

Consider the limit $u \to +\infty$ along the rays $v = \text{const}$. In this limit, $g(u) = \mathrm{e}^{-\delta_0}$, and r in (83) is a finite function of v, while $\partial r/\partial u$ in (82) turns into zero. It follows that the null line

$$\text{EH}: \qquad g(u) = \mathrm{e}^{-\delta_0} \tag{84}$$

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is the event horizon. *The collapse finishes with a true black hole.* The AH$^+$ continues up to $v = \infty$ and, in this limit, approaches the event horizon according to the law

$$
\text{AH}^+ : \quad
\begin{cases}
g(u) - \mathrm{e}^{-\delta_0} \ = \ \exp\left(-2\sqrt{\dfrac{2\lambda(-\delta_0)}{m_1{}^4}}\, v\right), & \delta_0 < 0, \quad v \to \infty \\[2ex]
g(u) - \mathrm{e}^{-\delta_0} \ = \ \sqrt{3}(2-\sqrt{3})\,\dfrac{m_1{}^4}{2\lambda}\dfrac{1}{v^2}, & \delta_0 = 0, \quad v \to \infty
\end{cases}
\tag{85}
$$

or, equivalently,

$$
\text{AH}^+ : \quad
\begin{cases}
v \ = \ \tfrac{1}{2}\,u, & \delta_0 < 0, \quad u \to +\infty \\[2ex]
v \ = \ \sqrt{p}\,u, \quad p = \dfrac{2-\sqrt{3}}{\sqrt{3}}, & \delta_0 = 0, \quad u \to +\infty.
\end{cases}
\tag{86}
$$

Along both the EH and AH$^+$,

$$
\text{EH, AH}^+ : \qquad r \to m_1 + \sqrt{2\lambda(-\delta_0)}, \qquad v \to \infty.
\tag{87}
$$

All outgoing light rays $u = \text{const.}$ with $g(u) > \mathrm{e}^{-\delta_0}$ cross the AH$^+$ and go out to the future null infinity ($\mathcal{I}^+$).

In terms of the retarded time $g(u)$, the metric extends beyond the event horizon and can be studied. The main facts are these. The AH$^-$ also continues up to $v = \infty$ and is asymptotically of the form

$$
\text{AH}^- : \quad
\begin{cases}
\mathrm{e}^{-\delta_0} - g(u) \ = \ 4\mathrm{e}^{-\delta_0}\sqrt{\dfrac{m_1{}^4(-\delta_0)}{2\lambda}}\dfrac{1}{v}, & \delta_0 < 0, \quad v \to \infty \\[2ex]
\mathrm{e}^{-\delta_0} - g(u) \ = \ \sqrt{3}(2+\sqrt{3})\,\dfrac{m_1{}^4}{2\lambda}\dfrac{1}{v^2}, & \delta_0 = 0, \quad v \to \infty
\end{cases}
\tag{88}
$$

but, along it,

$$
\text{AH}^- : \qquad r \to m_1 - \sqrt{2\lambda(-\delta_0)}, \qquad v \to \infty.
\tag{89}
$$

All outgoing light rays $u = \text{const.}$ with $g(u) < \mathrm{e}^{-\delta_0}$ cross the AH$^-$ and reach $v = \infty$. At $v = \infty$, all of them have the same finite value of r:

$$
g(u) = \text{const.} < \mathrm{e}^{-\delta_0}: \qquad r \to m_1 - \sqrt{2\lambda(-\delta_0)}, \qquad (\nabla r)^2 \to 0, \qquad v \to \infty.
\tag{90}
$$

Their end-points at $v = \infty$ make a line which will be called $\mathcal{C}$. All of the said light rays cross $\mathcal{C}$ but, for the metric to be obtained beyond $\mathcal{C}$, the data functions $m(v)$ and $q^2(v)$ need to be known for "$v > \infty$". There is nowhere to take these data from. $\mathcal{C}$ is a Cauchy horizon.

The chart u ends at the event horizon. The chart $g(u)$ continues farther but it has an end as well. It extends from the line L on which $g(u) = f(v)$ down to $g(u) = -\infty$. In the limit $g(u) \to -\infty$ along the rays $v = $ const., r in (83) is a finite function of v, while $\partial r/\partial g$ in (54) turns into zero:

$$\frac{\partial r(u,v)}{\partial g(u)} = -2m_1{}^2 \frac{f'(v)}{(f(v) - g(u))^2} \to 0 , \qquad g(u) \to -\infty . \tag{91}$$

The null line $g(u) = -\infty$ will be called $\mathcal{R}$. One can prove that this line is the light signal coming from the singularity. It puts an end to the present consideration. The rays $v = $ const. cross $\mathcal{R}$ but what is beyond $\mathcal{R}$ is also beyond the validity of semiclassical theory.

The full Penrose diagram is given in Fig. 2. The apparent horizon is almost closed in a sense that almost all outgoing light rays and almost all incoming ones cross it twice. The exceptions are the event horizon and the rays $v = $ const. in the classical interval of v ($v \leq v_{\text{crit}}$). The lines $\mathcal{R}$ and $\mathcal{C}$ bound the region of validity of the present solution.

The boundary $\mathcal{R}$ is caused by the lack of knowledge about the region of large curvature and does not present an unsolvable problem. It is only the limit of validity of semiclassical theory. If one has or will have a theory valid at small scale, then it will remove this boundary. For example, if one trusts the local terms of the effective action even despite the fact that their coefficients are uncertain, then the local polarization can be calculated [2]. It removes the singularity from the line $r = 0$ and shifts the apparent horizon away from this line. The apparent horizon then closes in the sector between the points 1 and 2 in Fig. 2.

The real problem is the unpredictability caused by the Cauchy horizon $\mathcal{C}$ because this boundary is in the region of validity of semiclassical theory and cannot be removed. It is important to emphasize the difference from the case of the Reissner–Nordstrom metric whose analytic continuation is readily obtained. In that case one deals with a *given* metric. When the metric is obtained by integrating the evolution equations from the initial state, as in the case of collapse, the Cauchy horizon is an insurmountable barrier. In a sense, it is a horizon for theoretical physics. The outgoing light rays pass through it to another universe, and there is no way to learn what will be with them next.

Appendix. Derivation of the modified data equations.

It suffices to consider the contributions of the s-mode ($\mathbf{\Psi}_0$) to the fluxes of $\mathcal{M}$ and Q^2 because only the s-mode contributes to the flux of Q^2 [1], and only the flux of Q^2 is significant when $\kappa = \mathcal{O}$. For these contributions one has [1]

$$-\partial_u \mathcal{M} = \langle (\partial_u \mathbf{\Psi}_0)^2 \rangle \Big|_{\mathcal{I}^+} - \xi \, \partial_{uu}^2 \langle \mathbf{\Psi}_0{}^2 \rangle \Big|_{\mathcal{I}^+} , \qquad (A.1)$$

$$\partial_u Q^2 = (1 - 4\xi) \, \partial_u \langle \mathbf{\Psi}_0{}^2 \rangle \Big|_{\mathcal{I}^+} . \qquad (A.2)$$

Here ξ is the parameter of the scalar-field equation [1] which in 4 dimensions is $1/6$, and in 2 dimensions is zero. Both expectation values in (A.1), (A.2) are expressed through the same spectral function [1]:

$$\langle (\partial_u \mathbf{\Psi}_0)^2 \rangle \Big|_{\mathcal{I}^+} = \frac{2}{(4\pi)^2} \int_0^\infty d\varepsilon_{\text{out}} \, I_0(\varepsilon_{\text{out}}, u) + \text{c.c.} , \qquad (A.3)$$

$$\langle \mathbf{\Psi}_0{}^2 \rangle \Big|_{\mathcal{I}^+} = \frac{2}{(4\pi)^2} \int_0^\infty \frac{d\varepsilon_{\text{out}}}{\mathrm{i}\varepsilon_{\text{out}}} \int_{-\infty}^u d\bar{u} \, I_0(\varepsilon_{\text{out}}, \bar{u}) \exp\left(\mathrm{i}\varepsilon_{\text{out}}(u - \bar{u})\right) + \text{c.c.} , \qquad (A.4)$$

$$I_0(\varepsilon_{\text{out}}, u) = \int_0^\infty d\varepsilon_{\text{in}} \, \dot{U}(u) \int_{-\infty}^{u+0} du' \, \varepsilon_{\text{in}} \dot{U}(u') \exp\left(\mathrm{i}(\Omega - \Omega')\right) , \qquad (A.5)$$

$$\Omega - \Omega' = \varepsilon_{\text{in}} \left(U(u) - U(u')\right) + \varepsilon_{\text{out}}(u - u') , \qquad (A.6)$$

$$\dot{U}(u) \equiv \frac{dU(u)}{du} = \exp\left(-\int_{-\infty}^u du \, \kappa\right) . \qquad (A.7)$$

The calculation needs to be done under condition (26).

In (A.5) make the replacement of the integration variables

$$y = \varepsilon_{\text{in}} \dot{U}(u) \frac{1}{\widetilde{\kappa}(u)} , \qquad x = \varepsilon_{\text{in}} \dot{U}(u') \frac{1}{\widetilde{\kappa}(u')} . \qquad (A.8)$$

For the needed Jacobian to emerge, $\widetilde{\kappa}(u)$ should satisfy Eq. (30), and, in order that the lower limit in x could be set equal to zero, $\widetilde{\kappa}(u)$ should satisfy the boundary condition (31). With the function $\widetilde{\kappa}(u)$ thus defined, one obtains

$$I_0(\varepsilon_{\text{out}}, u) = \widetilde{\kappa}(u) \left[1 + P\left(u, \mathrm{i}z, \frac{d}{d\mathrm{i}z}\right) \right] F(z) \qquad (A.9)$$

where

$$z = \frac{\varepsilon_{\text{out}}}{\widetilde{\kappa}(u)} , \qquad F(z) = \frac{2\pi z \mathrm{e}^{-\pi z}}{\mathrm{e}^{\pi z} - \mathrm{e}^{-\pi z}} , \qquad (A.10)$$

and the function P is defined as follows. The equation

$$\ln \frac{x}{y} = \int\limits_{u'}^{u} du'' \, \widetilde{\kappa}(u'') \qquad (A.11)$$

following from (A.8) and (30) should be solved with respect to the quantity

$$\widetilde{\kappa}(u)(u - u') = \ln \frac{x}{y} + f\left(u, \ln \frac{x}{y}\right) . \qquad (A.12)$$

The function P is expressed through f in (A.12) as

$$\begin{aligned}
P\left(u, \mathrm{i}z, \ln \frac{x}{y}\right) &= \exp\left(\mathrm{i}z f\left(u, \ln \frac{x}{y}\right)\right) - 1 \\
&= \sum_{k,p} h_{k,p}(u)(\mathrm{i}z)^k \left(\ln \frac{x}{y}\right)^{k+p} , \qquad k \geq 1 , \quad p \geq 1 \qquad (A.13)
\end{aligned}$$

and is a series of the form (A.13).

Only the real part of $I_0(\varepsilon_{\text{out}}, u)$, and, therefore, only the even p in the series (A.13) contribute to the expectation value (A.3). Upon the insertion of (A.9) and (A.13) with $p = 2n$ in the spectral integral (A.3), this integral boils down to

$$\int\limits_{0}^{\infty} dz \, (\mathrm{i}z)^k \left(\frac{d}{d\mathrm{i}z}\right)^{k+2n} F(z) = \mathrm{i}(-1)^k k! \left(\frac{d}{d\mathrm{i}z}\right)^{2n-1} F(z)\bigg|_{z=0} . \qquad (A.14)$$

The even powers of z in $F(z)$ drop out of this expression. Only the odd part of $F(z)$ contributes, and this odd part is

$$\frac{1}{2}\left(F(z) - F(-z)\right) = -\pi z . \qquad (A.15)$$

It follows that only the terms with $p = 2$ in the series (A.13) contribute to the expectation value (A.3), and this contribution can be calculated:

$$\langle (\partial_u \Psi_0)^2 \rangle \big|_{\mathcal{I}^+} = \frac{1}{48\pi}\left(\widetilde{\kappa}^2 - \frac{2}{\widetilde{\kappa}} \frac{d^2 \widetilde{\kappa}}{du^2} + \frac{3}{\widetilde{\kappa}^2}\left(\frac{d\widetilde{\kappa}}{du}\right)^2\right) . \qquad (A.16)$$

Eq. (30) for $\widetilde{\kappa}$ can now be used to obtain finally

$$\langle (\partial_u \Psi_0)^2 \rangle \big|_{\mathcal{I}^+} = \frac{1}{48\pi}\left(\kappa^2 + 2\frac{d\kappa}{du}\right) . \qquad (A.17)$$

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If this expression is inserted in (A.1) with $\xi = 0$, the result for $\partial_u \mathcal{M}$ will be precisely the one that the 2-dimensional effective action gives. This is a good check.

For the calculation of the expectation value (A.4), introduce in (A.4) the new integration variables

$$\gamma = \varepsilon_{\text{out}}(u - \bar{u}) , \qquad \sigma = \int_{\bar{u}}^{u} du'' \, \widetilde{\kappa}(u'') , \tag{A.18}$$

and, as discussed in Ref. [1], set $\widetilde{\kappa}(u) = \kappa(u) = 0$ for $u < u_0$. Then (A.4) will take the form

$$\langle \, \boldsymbol{\Psi}_0{}^2 \rangle \Big|_{\mathcal{I}^+} = \frac{2}{(4\pi)^2} \int_0^{\infty} \frac{d\gamma}{i\gamma} \, e^{i\gamma} \int_0^{\Gamma} d\sigma \left(1 + P\left(\bar{u}, i\bar{z}, \frac{d}{di\bar{z}}\right) \right) F(\bar{z}) + \text{c.c.} \tag{A.19}$$

where

$$\Gamma = \int_{u_0}^{u} du \, \widetilde{\kappa} , \qquad \bar{z} = \frac{\gamma}{\widetilde{\kappa}(\bar{u})(u - \bar{u})} , \tag{A.20}$$

and the denominator of $\bar{z}$ should be expressed through σ. This expression (with the obvious change of notation) is just (A.12). It is only important that

$$\bar{z} = O\left(\frac{\gamma}{\sigma}\right) \to 0 \quad \text{as} \quad \sigma \to \infty \tag{A.21}$$

since one only needs the asymptotics of (A.19) at $\Gamma \to \infty$. From the leading asymptotics, the contribution of P drops out entirely, and one obtains

$$\langle \, \boldsymbol{\Psi}_0{}^2 \rangle \Big|_{\mathcal{I}^+} = \frac{4}{(4\pi)^2} \left(\int_0^{\infty} d\gamma \, \frac{\sin\gamma}{\gamma} \right) F(0) \, \Gamma = \frac{1}{8\pi}\Gamma , \qquad \Gamma \to \infty \tag{A.22}$$

$$\partial_u \langle \, \boldsymbol{\Psi}_0{}^2 \rangle \Big|_{\mathcal{I}^+} = \frac{1}{8\pi} \, \widetilde{\kappa}(u) , \qquad \int_{u_0}^{u} du \, \widetilde{\kappa} \to \infty . \tag{A.23}$$

That $\Gamma \to \infty$ is the needed limit, i.e., that $\Gamma \to \infty$ follows from (26) can be seen from the integrated Eq. (30):

$$\exp\left(-\int_{u_0}^{u} du \, \widetilde{\kappa} \right) = \frac{\widetilde{\kappa}(u_0)}{\widetilde{\kappa}(u)} \exp\left(-\int_{u_0}^{u} du \, \kappa \right) . \tag{A.24}$$

Since $\widetilde{\kappa}(u_0) = O(1)$, this quantity vanishes in the limit (26) by virtue of the boundary condition (31). Eqs. (A.1), (A.2) with $\xi = 1/6$ and the expectation values inserted from (A.17) and (A.23) are the equations presented in the text.

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Figure captions

Fig.1. The region of strong field restricted to condition (43). The horizontal lines are $v = $ const. The vertical lines are $u = $ const. The three bold curves are AH^-, $B = 0$, and AH^+. The three broken curves make a line of extrema (minima) of $(\nabla r)^2$ along the rays $u = $ const. L is the border between the strong-field and weak-field regions.

Fig.2. Penrose diagram for the collapse spacetime. The bold lines are AH^-, EH, and AH^+. The double lines are the boundaries $\mathcal{R}$ and $\mathcal{C}$. The wavy lines mean the singularity. The light curves are the level lines of r: (a) $r = 2M$, (b) $r = 2M - |\mathcal{O}|$, (c) $r = m_1 + \sqrt{2\lambda(-\delta_0)}$, (d) $r = m_1 - \sqrt{2\lambda(-\delta_0)}$, (e) $r < m_1 - \sqrt{2\lambda(-\delta_0)}$.
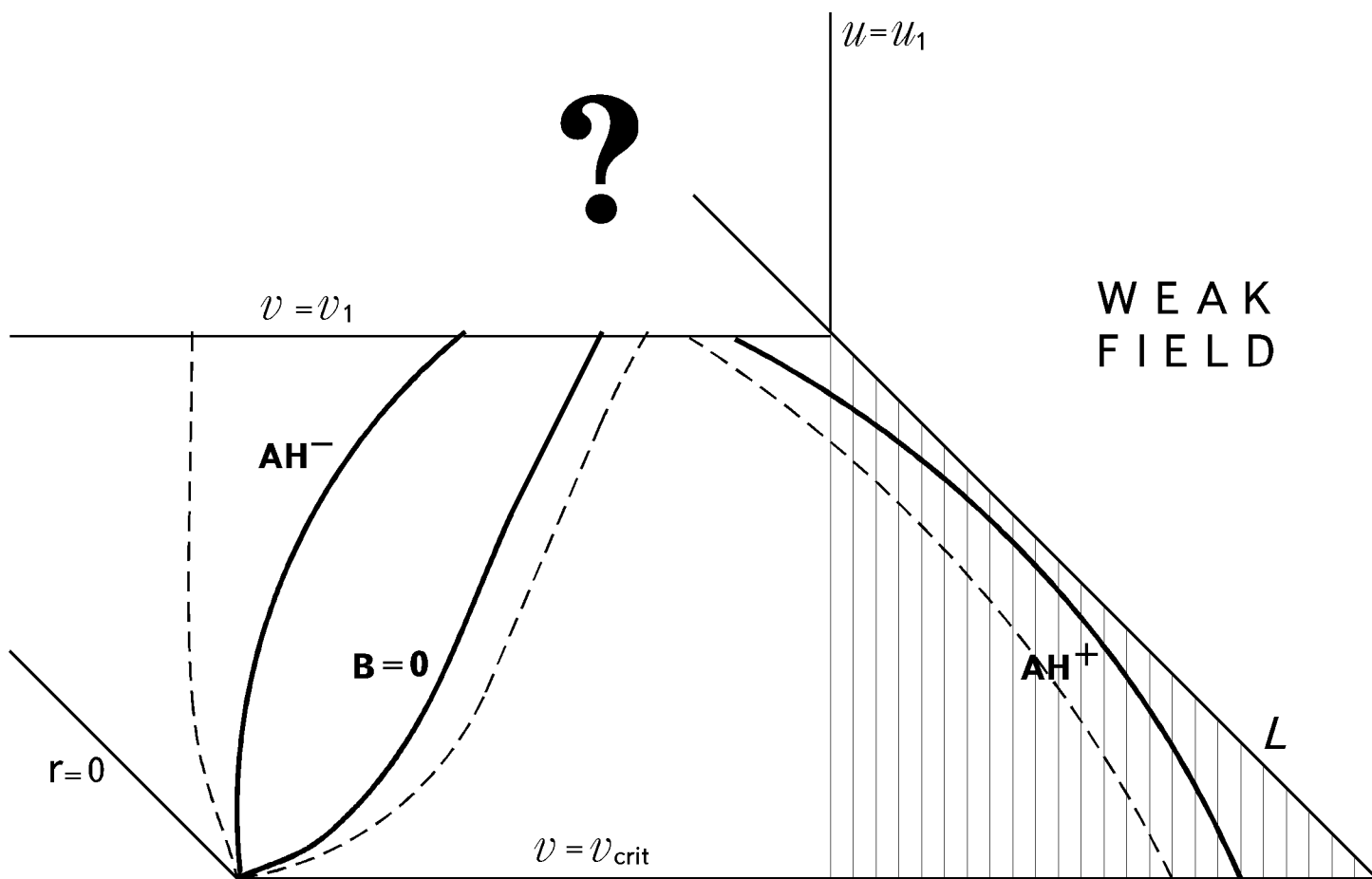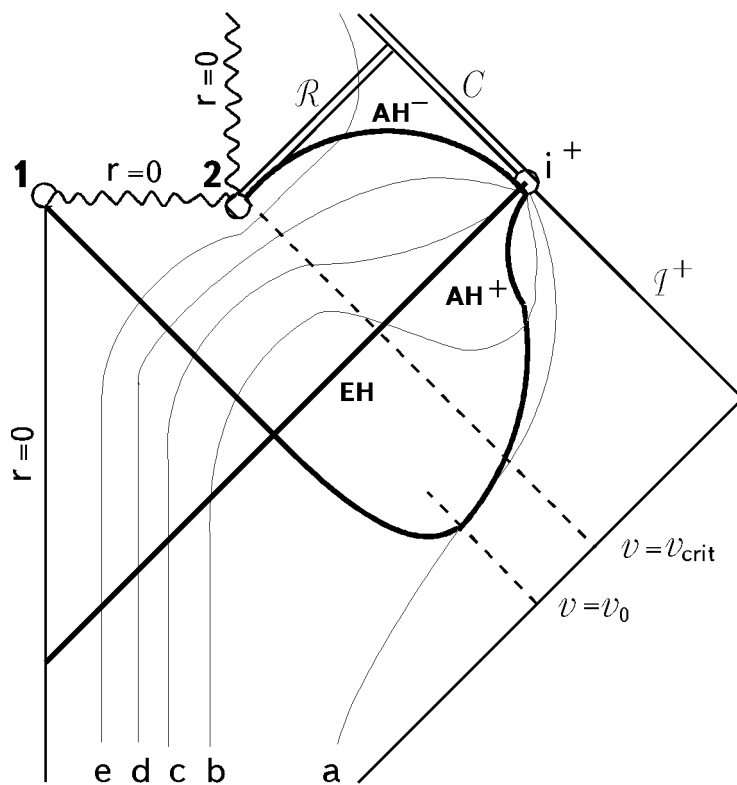
Fig.1

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Fig. 2